The Gauss' law

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Course:
Demo

Outline

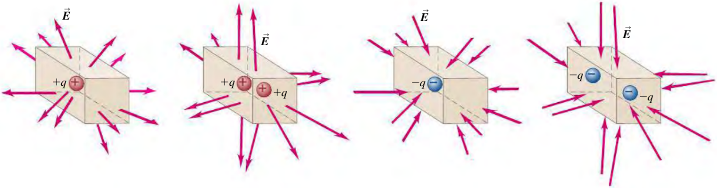
1. Introduction
2. Electric flux
3. Gauss' law
4. Applications
5. Conductors
6. Summary

Symmetries, electric fields & scope of Gauss' law

- Symmetries of systems as important tools for simplifying problems.
- Scope of Gauss' law:
  - A restatement of Coulomb's law.
  - Given any general distribution of charge, surround it with an **imaginary surface** that encloses the charge.
  - Look at the electric field at various points on this surface.
  - Establish a relationship between the field at all the points on the surface and the total charge enclosed within the surface.
  - A tremendously useful relationship!
- If the electric-field pattern is known in a given region, what can we determine about the charge distribution in that region?

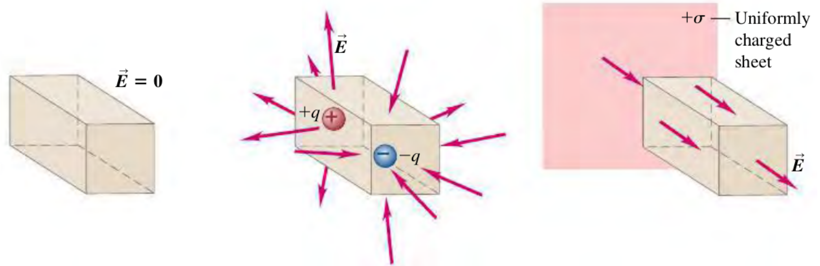
Electric flux I: Concepts

- "Flux" comes from a Latin word, meaning "flow".
- If the electric-field vectors point out of a surface, we say that there is an **outward electric flux**.
- If the electric-field vectors point into a surface, we say that there is an **inward electric flux**.
- More charges inside a closed surface cause a higher flux.
- Positive (+ve) charges inside a closed surface cause an outward flux.
- Negative (−ve) charges inside a closed surface cause an inward flux.
- What happens when there's no **net charge** inside?



Electric flux II: Charge dependence

- In the absence of charges, $\vec{E} = 0$ everywhere, thus the flux is zero.
- When the net charge is zero, *i.e.* $+q + (-q) = 0$, the number of field-lines moving into and out of the closed surface is the same. Thus, the inward and outward fluxes cancel each other out; resulting in a net flux of zero.
- In the presence of an external field, again the number of field-lines moving into and out of the closed surface is the same. Thus, the inward and outward fluxes cancel out to result in a net flux of zero.



Electric flux III: Observations

- Whether there is a net outward or inward electric flux through a closed surface depends on the sign of the enclosed net-charge.
- Charges outside a closed surface do not give a net electric flux through the closed surface.
- The net electric flux is directly proportional to the net amount of charge enclosed within the surface, but is otherwise independent of the size of the closed surface.
- These observations are a qualitative statement of Gauss' law.

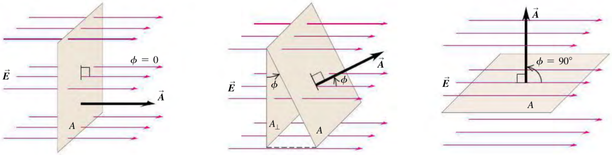
Electric flux IV: Formulation

- Flux through a surface can be defined as the product of the average perpendicular component of $\vec{\boldsymbol{E}}$ and the area of the surface: $\vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{S}}$.
- For the total flux through a closed surface, the individual fluxes through the different faces need to be summed over:

$$\Phi = \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{A}} = \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{S}}_1 + \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{S}}_2 + \ldots + \vec{\boldsymbol{E}} \cdot \vec{\boldsymbol{S}}_n \ .$$

- For a non-uniform electric field or a curved surface, we divide A into many small elements dA, each of which has a unit vector $\hat{\boldsymbol{n}}$ perpendicular to it and a vector area $d\vec{\boldsymbol{A}} = \hat{\boldsymbol{n}}\ dA$ to obtain:

$$\Phi = \oint_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} \ .$$

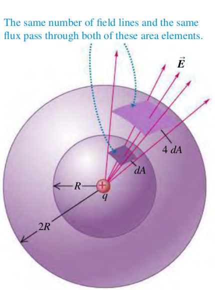


Gauss' law I: Introduction

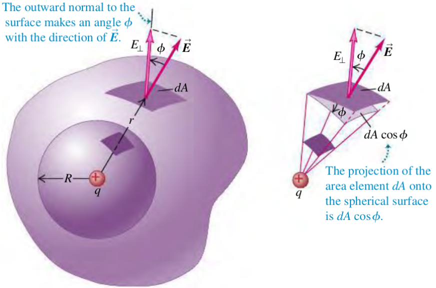
Statement

The total electric flux through any closed surface is equal to the net electric charge enclosed by the surface, divided by ϵ_0.

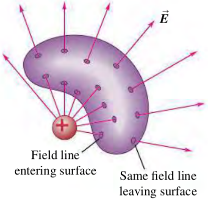
$$\Phi = \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\epsilon_0}$$



$$\Phi = \frac{q}{\epsilon_0}$$



$$\Phi = \frac{q}{\epsilon_0}$$

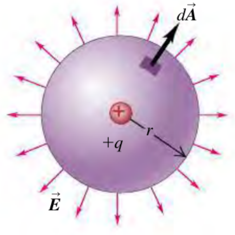


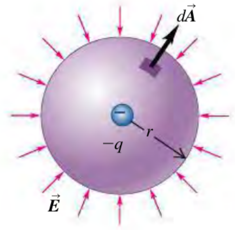
$$\Phi = 0$$

Gauss' law II: Problem-solving

- **Identify** the relevant concepts:
  Gauss' law is most useful when the charge distribution has spherical, cylindrical or planar symmetry.
- **Set up** the problem using the following steps:
  - List the known and unknown quantities.
  - Identify the target variable.
  - Select the appropriate closed, imaginary Gaussian surface.
- **Execute** the solution as follows:
  - Determine the size and placement of your Gaussian surface.
  - Evaluate the integral $\oint_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}}$.
  - Use Gauss' law to obtain the target variable.
- **Evaluate** your answer:
  If your result is a function that describes how the magnitude of the electric field varies with position, ensure that it makes sense.

Applications I: Point charge





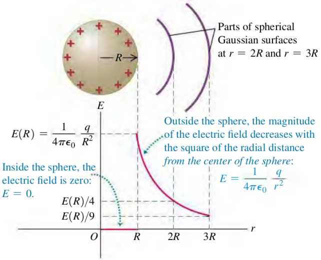
Calculation of electric field, given:

- net enclosed charge $= \pm q$,
- radius of Gaussian sphere $= r$ &
- $\vec{\boldsymbol{E}} \parallel d\vec{\boldsymbol{A}}$ everywhere.

$$\therefore \oint_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = 4\pi r^2 E = \pm \frac{q}{\epsilon_0}$$

$$\implies \boxed{E(r) = \pm \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}}$$

Applications II: Spherical conductor



Calculation of electric field:

I. Inside the sphere, given:-

- net enclosed charge $= 0$ &
- $r < R$.

$$\therefore \oint_S \vec{E}\cdot d\vec{A} = 4\pi r^2 E = 0$$

$$\implies \boxed{E = 0}$$

II. At the surface, given:-

- net enclosed charge $= +q$,
- $r = R$ &
- $\vec{E} \parallel d\vec{A}$ everywhere.

$$\therefore \oint_S \vec{E}\cdot d\vec{A} = 4\pi R^2 E = \frac{q}{\epsilon_0}$$

$$\implies \boxed{E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}}$$

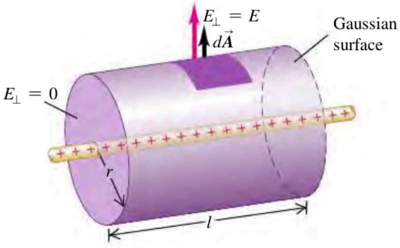
III. Outside the sphere, given:-

- net enclosed charge $= +q$,
- $r > R$ &
- $\vec{E} \parallel d\vec{A}$ everywhere.

$$\therefore \oint_S \vec{E}\cdot d\vec{A} = 4\pi R^2 E = \frac{q}{\epsilon_0}$$

$$\implies \boxed{E(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}}$$

Applications III: Uniform line charge



Calculation of electric field, given:

- linear charge density $= +\lambda$,
- cylindrical Gaussian surface, with radius r and arbitrary length l, coaxial with wire and with ends perpendicular to wire,
- $\vec{E} \parallel d\vec{A}$ along curved surface &
- $\vec{E} \perp d\vec{A}$ along plane surfaces.

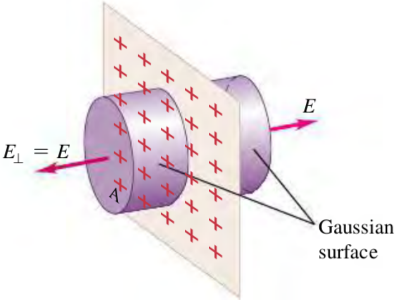
$$\therefore \oint_S \vec{E} \cdot d\vec{A} = \int_C \vec{E} \cdot d\vec{A} + \int_P \vec{E} \cdot d\vec{A}$$
$$= 2\pi r l E + 0$$
$$= 2\pi r l E \; .$$

Now, $Q_{\text{encl}} = \lambda l$. Thus:

$$2\pi r l E = \frac{\lambda l}{\epsilon_0}$$

$$\Longrightarrow \boxed{E(r) = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}}$$

Applications IV: Infinite plane sheet of charge



Calculation of electric field, given:

- surface charge density $= +\sigma$,
- cylindrical Gaussian surface with ends of area A and with axis $\perp$ to sheet of charge,
- $\vec{\boldsymbol{E}} \perp d\vec{\boldsymbol{A}}$ along curved surface &
- $\vec{\boldsymbol{E}} \parallel d\vec{\boldsymbol{A}}$ along plane surfaces.

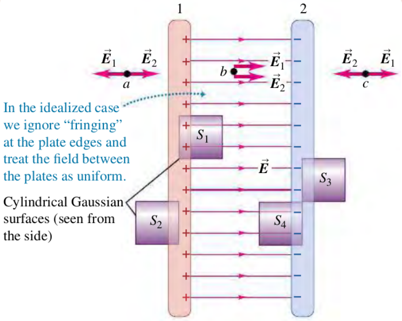
$$\begin{aligned}\therefore \oint_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} &= \int_C \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} + \int_P \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} \\ &= 0 + \int_{P_1} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} + \int_{P_2} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} \\ &= 2EA\ .\end{aligned}$$

Now, $Q_{\text{encl}} = \sigma A$. Thus:

$$2EA = \frac{\sigma A}{\epsilon_0}$$

$$\Longrightarrow \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

Applications V: Charged parallel conducting plates



Calculation of electric field, given:

- surface charge density $= \pm\sigma$,
- cylindrical Gaussian surfaces with ends of area A and with one end of each lying within a plate,
- $\vec{E} \perp d\vec{A}$ along curved surfaces &
- $\vec{E} \parallel d\vec{A}$ along plane surfaces.

Consider S_1:

$$\oint_S \vec{E} \cdot d\vec{A} = EA = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$\Longrightarrow \boxed{E = \frac{\sigma}{\epsilon_0}}$$

Now with S_4:

$$\oint_S \vec{E} \cdot d\vec{A} = -EA = \frac{Q_{\text{encl}}}{\epsilon_0} = -\frac{\sigma A}{\epsilon_0}$$

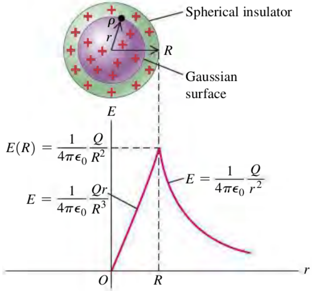
$$\Longrightarrow \boxed{E = \frac{\sigma}{\epsilon_0}}$$

However, with S_2 & S_3:

$$\oint_S \vec{E} \cdot d\vec{A} = EA = \frac{Q_{\text{encl}}}{\epsilon_0} = 0$$

$$\Longrightarrow \boxed{E = 0}$$

Applications VI: Uniformly charged spherical insulator



Calculation of electric field:

I. Inside the sphere, given:-

- volume charge density $= +\rho$ &
- $r < R$.

$$\therefore \oint_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = 4\pi r^2 E$$

Now, $Q_{\text{encl}} = \rho V$ gives:

$$4\pi r^2 E = \frac{Q}{\epsilon_0 \left(\frac{4}{3}\pi R^3\right)} \left(\frac{4}{3}\pi r^3\right)$$

$$\Longrightarrow \boxed{E(r) = \frac{1}{4\pi\epsilon_0} \frac{Qr}{R^3}}$$

II. Outside the sphere, given:-

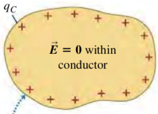
- net enclosed charge $= +Q$,
- $r > R$ &
- $\vec{\boldsymbol{E}} \parallel d\vec{\boldsymbol{A}}$ everywhere.

$$\therefore \oint_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = 4\pi r^2 E = \frac{Q}{\epsilon_0}$$

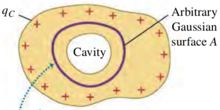
$$\Longrightarrow \boxed{E(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}}$$

Conductors I: Cavities

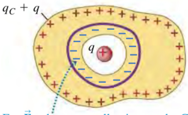
- For **uncharged conductor with empty cavity** inside, Gauss' law mandates that net charge on surface of cavity be zero, because $\vec{E} = 0$ everywhere on the Gaussian surface.
- For **uncharged conductor with charge $+q$ inside cavity**, Gauss' law mandates that charge $+q$ appear on its outer surface, because $\vec{E} = 0$ everywhere on the Gaussian surface.
- For **charged conductor with charge $+q$ inside cavity**; carrying charge q_C; total charge on outer surface is $q_C + q$.
- Anything inside conductor's cavity is **electrostatically shielded** from fields outside—used to build **Faraday cages**.



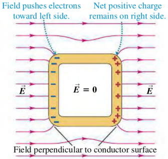
The charge q_C resides entirely on the surface of the conductor. The situation is electrostatic, so $\vec{E} = 0$ within the conductor.



Because $\vec{E} = 0$ at all points within the conductor, the electric field at all points on the Gaussian surface must be zero.

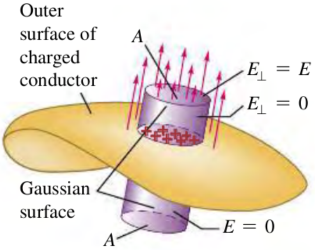


For $\vec{E}$ to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$.



Conductors II: Surface charge

- The $\vec{E}$ field at a point just outside a conductor is directly proportional to the surface charge density σ at that point.
- At the surface of spherical conductors: $E = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}$.
- Surface charge density defined as: $\sigma = \frac{\text{Total charge}}{\text{Total surface area}}$.
- Thus, at the surface of spherical conductors: $E = \frac{\sigma}{\epsilon_0}$.
- Similar results obtained for charged plates.
- In general, at the surface of a charged conductor:



- $\vec{E} \parallel \vec{A}$ for outer plane surface,
- $\vec{E} \perp$ outer curved surface &
- $\vec{E} = 0$ inside conductor.

$$\therefore \boxed{E = E_\perp = \frac{\sigma}{\epsilon_0}}$$

Electric flux

Electric flux is a measure of the "flow" of electric field through a surface. It is equal to the product of an area element and the perpendicular component of $\vec{\boldsymbol{E}}$, integrated over a surface.

$$\Phi = \int_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \int_S E_\perp dA$$

Gauss' law

Gauss' law states that the total electric flux through a closed surface, which can be written as the surface integral of the component of $\vec{\boldsymbol{E}}$ normal to the surface, equals a constant times the total charge Q_{encl} enclosed by the surface. It's logically equivalent to Coulomb's law, but its use greatly simplifies problems with a high degree of symmetry.

$$\Phi = \oint_S \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \oint_S E_\perp dA = \frac{Q_{\text{encl}}}{\epsilon_0}$$

When excess charge is placed on a solid conductor and is at rest, it resides entirely on the surface. $\vec{\boldsymbol{E}} = 0$ everywhere inside the conductor.

Electric fields of various symmetric charge distributions

Charge distribution	Position	Electric field
Single point: q	Distance r from q	$E(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$
Conducting sphere: R, q	Outside sphere, $r > R$ Inside sphere, $r < R$	$E(r) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ $E = 0$
Infinite wire: λ	Distance r from wire	$E(r) = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}$
Insulating sphere: R, Q	Outside sphere, $r > R$ Inside sphere, $r < R$	$E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$ $E(r) = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}$
Infinite sheet: σ	Any point	$E = \frac{\sigma}{2\epsilon_0}$
Conducting plates: $\pm\sigma$	Any point between plates	$E = \frac{\sigma}{\epsilon_0}$
Conductor: σ	At surface of conductor	$E = \frac{\sigma}{\epsilon_0}$