The phenomenology of heavy-ion collisions

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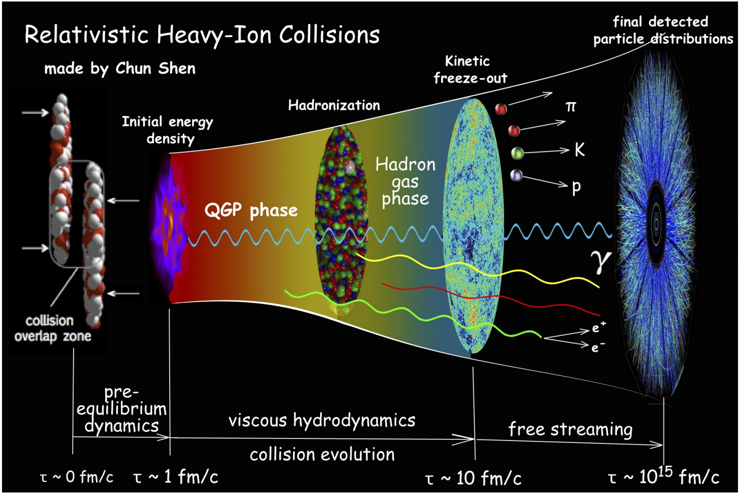
Outline

HIC: Schematic



Source: Chun Shen, The Ohio State University

HIC: Stages of evolution I

Pre-equilibrium:

- Beginning in the immediate aftermath of the collision between the projectile and target nuclei, this stage can be described by binary collisions of two nucleons.
- The kinetic energy of these collisions is then transferred to the produced particles and fields; both of the partonic (quark) and hadronic type.
- Following this, the produced particles and fields start interacting with the reaction products of other constituent collisions, thereby producing more particles and resulting in a system with high enough energy and particle density for the second stage to set in.

HIC: Stages of evolution II

Equilibrium expansion:

- After the initial product particles scatter multiple times, the system reaches a local thermal equilibrium.
- It exhibits collective behaviour and can be characterised by intensive quantities like pressure, energy density and particle-number density.

HIC: Stages of evolution III

Decoupling:

- Following the equilibrium expansion, the system dilutes to an extent that local or chemical equilibrium conditions are no longer fulfilled.
- The hadrons are formed at these stage, and having decoupled from the system, they start flying into the detectors to be measured.

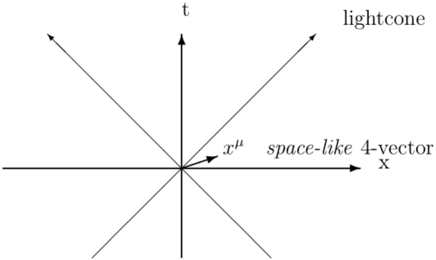
HIC: Transport models vs. effective models

1. The pre-equilibrium & decoupling stages (I & III) deal predominantly with particles, since the density; at ground zero; is relatively low - w.r.t. the equilibrium stage (II) - but the temperature and energy are high. This makes relativistic, microscopic transport models perfect for tracking these two stages.
2. The equilibrium stage (II) predominantly deals with fields. Both the temperature and the density are high. This provides conditions germane to the application of hydrodynamics, which requires an equation-of-state (EOS) to operate. This EOS is provided by effective-Lagrangian, thermodynamic models.

Kinetic theory: Microscopic quantities I

These **phase-space variables** are defined using unit convention $c = k_B = 1$ & metric $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$

- Spacetime
  - The 3 spatial co-ordinates and time span a 4-D space, called spacetime or position-space.
  - The coordinates are composed of time, t, and the position 3-vector, $\vec{r}$ as $x^\mu = (t, \vec{r}) = (x^0, \vec{x})$.



Kinetic theory: Microscopic quantities II

- Four-momentum
  - Defined as $p^\mu = (p^0, \vec{p})$, with $p^0 = \sqrt{|\vec{p}|^2 + m^2}$; since $p^0 = E$ at relativistic energies.
  - Transforms as a time-like vector (*i.e.*, $q^\mu q_\mu > 0$), with the normalisation:

$$(p^0)^2 - |\vec{p}|^2 = \sum_\mu p^\mu p_\mu = p^\mu p_\mu = m^2 \ . \tag{1}$$

  - The covariant and contravariant components are related as:

$$p_\mu = g_{\mu\nu} p^\nu \tag{2}$$

$$\implies p^\mu = (p^0, \vec{p}) \ \& \ p_\mu = (p^0, -\vec{p}) \ . \tag{3}$$

Kinetic theory: Microscopic quantities III

- Four-velocity
  - Time-like unit vector, pointing in the direction-of-motion, with components $(u^0, \vec{v})$; where $\vec{v} = \vec{p}/p^0$.
  - Normalisation and definition:

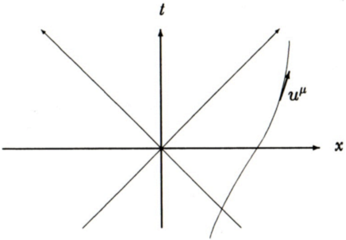
$$u^\mu u_\mu = +1 \; , \tag{4}$$

$$u^\mu = (\gamma, \gamma\vec{v}) \;\&\; u_\mu = (\gamma, -\gamma\vec{v}) \; . \tag{5}$$

  - $\gamma = 1/\sqrt{1 - |\vec{v}|^2}$ is the Lorentz factor.
  - Proper time of an object, moving with 4-velocity u^μ is $\tau = t/\gamma$.

Kinetic theory: Microscopic quantities IV

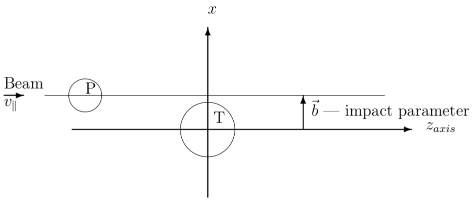
- World-line of a particle in space-time



- Space-like unit 4-vectors: $\Lambda^\mu \Lambda_\mu = -1$.
- Time-like unit 4-vectors: $\Lambda^\mu \Lambda_\mu = +1$.
- Space- and time-like vectors can't be transformed into each other by a proper Lorentz transformation.

Kinetic theory: Microscopic quantities V

- The nuclear-collision co-ordinate system
  - In particle and nuclear physics, it is practical to introduce a special co-ordinate system, where the spatial z-axis is parallel to the beam of the accelerator.
  - In general, non-central collisions, the 3-vector connecting the centres of a beam-particle and a target-particle points out in another direction. The component of this vector orthogonal to the beam is the impact-vector, $\vec{b}$, which is a two-dimensional vector. The direction of this vector is denoted usually as the x-direction. These two axes, x and z, span the so called reaction plane of a given collision $[x, z]$.



Kinetic theory: Microscopic quantities VI:- Rapidity

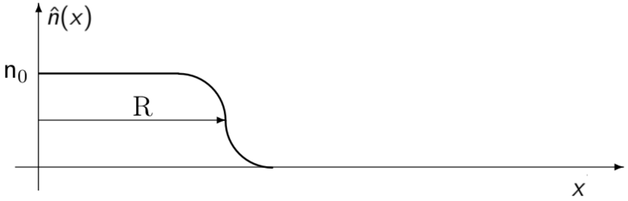
- Rapidity
  - Defined as the generalised velocity:

$$y = \tanh^{-1} v_{\parallel} = \tanh^{-1}\left(\frac{p_{\parallel}}{p^0}\right) = \frac{1}{2}\ln\left(\frac{p^0 + p_{\parallel}}{p^0 - p_{\parallel}}\right) . \qquad (6)$$

  - For small velocities: $y \approx v_{\parallel}$.
  - If a particle is moving after the collision into some direction $\vec{v}$, its phase space position is given by the coordinates $(y, p_{\perp}/m)$.
  - The velocity is limited to 1 (c); but rapidity range is $(-\infty, \infty)$.
  - Transformation properties, transverse mass and pseudo-rapaidity:
    - If y_1 is the rapidity of a particle in frame K_1, and y_2 is the rapidity of frame K_1 w.r.t. K_2, then $y = y_1 + y_2$ is the particle's rapidity in K_2; *i.e.*; rapidity is additive under a Lorentz transformation.
    - Transverse mass:
      $m_{\perp} = \sqrt{m^2 + p_{\perp}^2} = E/\cosh(y) = p^0/\cosh(y)$.
    - Pseudo-rapidity, η: for $E \to \infty$, $y \to \eta = \ln(\cot\Theta/2)$, where Θ is the polar angle of the emitted particle.

Kinetic theory: Macroscopic quantities I

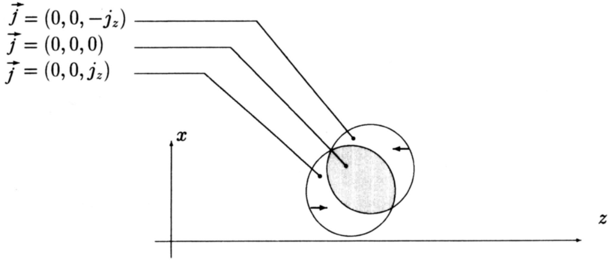
- Local particle density
  - A function of spacetime: $\hat{n} \equiv \hat{n}(t, \vec{r}) \equiv \hat{n}(x^{\mu}) \equiv \hat{n}(x)$.



  - Number of particles in a 3-volume (3-D hypersurface of 4-D spacetime) element $\Delta^3 x$: $\hat{N} = \hat{n}(x)\Delta^3 x$.
  - $\hat{n}$ is a scalar, but is not Lorentz invariant, since the 3-volume element $\Delta^3 x$ is not Lorentz invariant.
  - $\hat{N}$ is an invariant scalar.

Kinetic theory: Macroscopic quantities II

- Local particle current
  - Particle current across unit area per unit time: $\vec{j} \equiv \vec{j}(x)$.
  - When two equally sized nuclei collide; in their centre-of-mass system; the currents are directed along the beam (z-axis) and they are opposite to each other in the projectile and target regions. In the overlap region (shaded area), the two currents cancel each other out and the resulting current vanishes (if side-ward flow and squeeze-out of the matter are neglected).



Kinetic theory: Macroscopic quantities III

- Particle 4-current
  - A 4-vector constructed by combining the local particle density and the local particle current as:

$$N^{\mu}(x) = (\hat{n}(x), \vec{j}(x)) \ . \tag{7}$$

- Particle distribution
  - The distribution of particles, $f(x, p)$, in the 6-D $(x^0, x^1, x^2, p^0, p^1, p^2)$ space, or phase-space.
  - Gives the no. of particles, N, in a phase-space volume element as:

$$N = f(x, p) \ \Delta^3 x \ \Delta^3 p \ . \tag{8}$$

Kinetic theory: Macroscopic quantities IV

$f(x,p)$ & $N^\mu(x)$ can be used to obtain:

$$\hat{n}(x) = \int d^3p \; f(x,p) \; , \tag{9}$$

$$\vec{j}(x) = \int d^3p \; \vec{v} \; f(x,p) \quad \& \tag{10}$$

$$\boxed{N^\mu(x) = \int \frac{d^3p}{p^0} \; p^\mu \; f(x,p)} \tag{11}$$

It can be shown (cf.: Csernai's book) that $\int \frac{d^3p}{p^0}$ and $f(x,p)$ are both invariant scalars $\implies$ N^μ transforms like a 4-vector, since p^μ is a 4-vector.

Kinetic theory: Macroscopic quantities V:- $T^{\mu\nu}$ I

- The energy-momentum tensor
  - A macroscopic, 16-component tensor; characterising the matter.
  - Components:

$$\text{Energy density}: T^{00}(x) = \int d^3p \; p^0 \; f(x,p) = \int \frac{d^3p}{p^0} \; p^0 \; p^0 \; f(x,p)$$

$$\text{Energy flow}: T^{0i}(x) = \int d^3p \; p^0 \; \vec{v} \; f(x,p) = \int \frac{d^3p}{p^0} \; p^0 \; \vec{p} \; f(x,p)$$

$$\text{Mom. density}: T^{i0}(x) = \int d^3p \; \vec{p} \; f(x,p) = \int \frac{d^3p}{p^0} \; \vec{p} \; p^0 \; f(x,p)$$

$$\text{Pressure tensor}: T^{ij}(x) = \int d^3p \; \vec{p}\cdot\vec{v} \; f(x,p) = \int \frac{d^3p}{p^0} \; \vec{p}\cdot\vec{p} \; f(x,p)$$

$$\boxed{T^{\mu\nu}(x) = \int \frac{d^3p}{p^0} \; p^\mu p^\nu \; f(x,p)} \tag{12}$$

Kinetic theory: Macroscopic quantities VI:- $T^{\mu\nu}$ II

- Properties of the energy-momentum tensor:
  - It is the second moment of the distribution function $f(x, p)$.
  - It is symmetric: $T^{\mu\nu} = T^{\nu\mu}$.
  - It does not include the particles' fields and potential energies.
  - It only includes their rest-masses and kinetic energies.
  - For particle interactions via fields, the contribution of the fields need to be added to the energy-momentum tensor.

Kinetic theory: Macroscopic quantities VII:- u^μ, $\Delta^{\mu\nu}$ & LRF

- The flow vel. of a medium, u^μ, is a unit vector parallel to the world line of the particles or the direction of the energy flow.
- The orthogonal projection operator:

$$\Delta^{\mu\nu} = g^{\mu\nu} - \frac{u^\mu u^\nu}{(u^\nu u_\nu)} \ ,$$

  projects any 4-vector onto the 3-D hypersurface orthogonal to u^μ.
- The Local Rest Frame (LRF) is defined as the frame-of-reference where $u^\mu = (1, 0, 0, 0) = u^\mu_{LRF}$.
- In the LRF:
  - $\Delta^{\mu\nu}_{LRF} = \Delta^{LRF}_{\mu\nu} = \mathrm{diag}(0, -1, -1, -1)$,
  - $\Delta^{\mu}_{\nu\ (LRF)} = \mathrm{diag}(0, 1, 1, 1)$,
  - $N^i = 0 \implies \Delta_{\mu\nu} N^\mu = 0$ &
  - $T^{0i} = T^{i0} = 0 \implies \Delta_{\mu\alpha} T^{\mu\nu} u_\nu = 0$.

Kinetic theory: Macroscopic quantities VIII

- Invariant scalar density: $n = N^{\mu} u_{\mu} \implies n = N^0_{LRF}$.
- Invariant energy density: $\epsilon = u_{\mu} T^{\mu\nu} u_{\nu} \implies \epsilon = T^{00}_{LRF}$.
- Pressure tensor: $P^{\mu\nu} = \Delta^{\mu}_{\alpha} T^{\alpha\beta} \Delta^{\nu}_{\beta} \implies P = \delta^{ij} T^{ij}_{LRF}$ (pressure).
- $T^{\mu\nu}$ & N^{μ} in the LRF:

$$T^{\mu\nu\,(0)}_{LRF} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} , \; N^{\mu\,(0)}_{LRF} = (n, 0, 0, 0) \; ; \qquad (13)$$

where the index 0 denotes the non-dissipative component, used in ideal hydrodynamics.

Kinetic theory: Mixtures

- Particle distribution function in multi-component systems: $f_k(x, p_k)$, $k = 1, 2, 3...\mathcal{N}$; with rest masses: $p_k^\mu p_{\mu k} = m_k^2$.
- Particle 4-currents:

$$N_k^\mu(x) = \int \frac{d^3p_k}{p_k^0} \, p_k^\mu \, f(x, p_k) \implies N^\mu = \sum_k N_k^\mu \, . \tag{14}$$

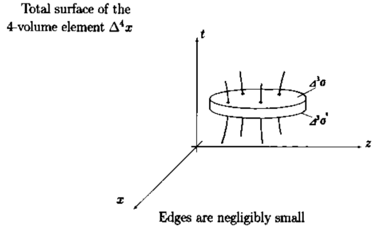
- Energy-momentum tensors:

$$T_k^{\mu\nu}(x) = \int \frac{d^3p_k}{p_k^0} \, p_k^\mu p_k^\nu \, f(x, p_k) \implies T^{\mu\nu} = \sum_k T_k^{\mu\nu} \, . \tag{15}$$

- Conserved charge-currents (if the particle no. is not conserved):

$$Q_k^\mu(x) = \int \frac{d^3p_k}{p_k^0} \, q_k^\mu \, p_k^\mu \, f(x, p_k) \implies Q^\mu = \sum_k Q_k^\mu \, . \tag{16}$$

Relativistic transport: The continuity equation



For negligibly small edges of the 4-volume $\Delta^4 x$:

$$\int_{\Delta^3\sigma}\int_{\Delta^3 p} d^3\sigma_\mu \, \frac{d^3p}{p^0} \, p^\mu \; f(x,p) \tag{17}$$

$\rightarrow$

$$\oint_{\Delta^3\sigma}\int_{\Delta^3 p} d^3\sigma_\mu \, \frac{d^3p}{p^0} \, p^\mu \; f(x,p) = 0. \tag{18}$$

Using Gauss' theorem &

$$\frac{\partial}{\partial x^\mu}(p^\mu f) = \partial_\mu(p^\mu f) = p^\mu \partial_\mu f \;\; ,$$

with eqn. (18):

$$\oint_{\Delta^3\sigma} d^3\sigma_\mu A^\mu = \int_{\Delta^4 x} \frac{\partial}{\partial x^\mu} A^\mu d^4x$$

$\rightarrow$

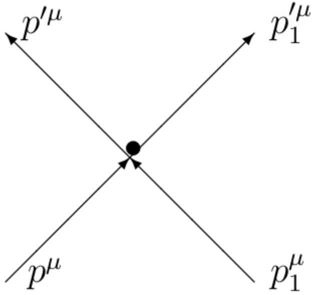
$$\int_{\Delta^4 x}\int_{\Delta^3 p} d^4x \frac{d^3p}{p^0} \; p^\mu \partial_\mu f(x,p) = 0; \tag{19}$$

the continuity equation looks like:

$$\boxed{p^\mu \partial_\mu f(x,p) = 0} \tag{20}$$

Relativistic transport: The Boltzmann transport equation I

In the presence of external forces, *i.e.*, collisions,



the continuity eqn. looks like:

$$p^{\mu}\partial_{\mu} f(x,p) = \mathcal{C}(x,p) \ .$$

Here $\mathcal{C}$ is the collision integral; dependent on:

- the number of nucleons (particles) around $\vec{p}$: $\Delta^3 p f(x,p)$,
- the number of nucleons (particles) around $\vec{p}_1$: $\Delta^3 p_1 f(x,p_1)$ and
- the final state and configuration volume intervals: $\Delta^3 p', \Delta^3 p_1'$ & $\Delta^4 x$;

with the proportionality factor (transition rate $= W$)

$$\frac{W(p,p_1|p',p_1')}{p^0 p_1^0 p'^0 p_1'^0}$$

Relativistic transport: The Boltzmann transport equation II

The most general, multi-component, relativistic form of the BTE is:

$$p_k^\mu \partial_\mu f(x, p_k) = \sum_{l=1}^{\mathcal{N}} \mathcal{C}_{kl}(x, p_k) \qquad (21)$$

where it can be shown (cf.: Csernai book), for a general $k + l \rightarrow i + j$ collision:

$$\mathcal{C}_{kl}(x, p_k) = \frac{1}{2} \sum_{i,j=1}^{\mathcal{N}} \int \frac{d^3 p_l}{p_l^0} \frac{d^3 p_i}{p_i^0} \frac{d^3 p_j}{p_j^0} \left[f_i f_j W_{ij|kl} - f_k f_l W_{kl|ij} \right] \ . \qquad (22)$$

Properties of the transition rate: $W_{kl|ij} = W_{kl|ji}$ & $W_{kl|ij} = W_{ij|kl}$

Relativistic transport: Conservation laws from the BTE

The full BTE can be used to obtain the conservation laws:
For a microscopic quantity Ψ_k determined by the particle type (k), position (x) and momentum (p_k); which is conserved in a binary collision $k + l \rightarrow i + j$; one has (cf.: Csernai book)

$$\sum_{k,l=1}^{\mathcal{N}} \int \frac{d^3 p_k}{p_k^0} \Psi_k \mathcal{C}_{kl}(x, p_k) = 0 \tag{23}$$

1. From eqns. (14), (21) & (23), particle no. conservation: $\partial_\mu N^\mu = \sum_k \partial_\mu N_k^\mu = 0$, with $\Psi_k = 1$.
2. From eqns. (16), (21) & (23), charge conservation: $\partial_\mu Q^\mu = \sum_k \partial_\mu Q_k^\mu = 0$, with $\Psi_k = q_k$.
3. From eqns. (15), (21) & (23), energy-momentum conservation: $\partial_\mu T^{\mu\nu} = \sum_k \partial_\mu T_k^{\mu\nu} = 0$, with $\Psi_k = p_k^\nu$.

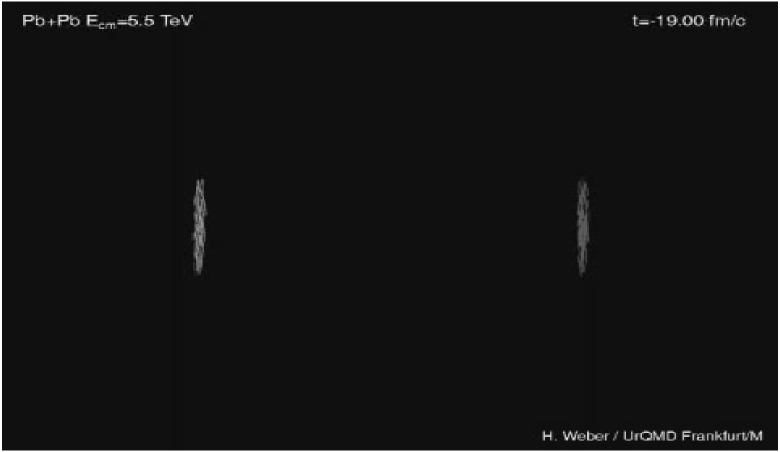
Relativistic transport: The BTE epilogue

- The conservation laws do not form a closed set of equations.
- $T^{\mu\nu}$ and N^{μ} should be defined, too.
- In the transport theory, this is done through the distribution function, which is known only if the solution to the BTE is known.
- Thus, within the transport theory, these equations do not provide the solution to a dynamical problem.
- A sol. can be found by theorising the functional form of $f(x, p_k)$, or
- One can start with an effective-solution of the BTE...

The UrQMD model: Basics

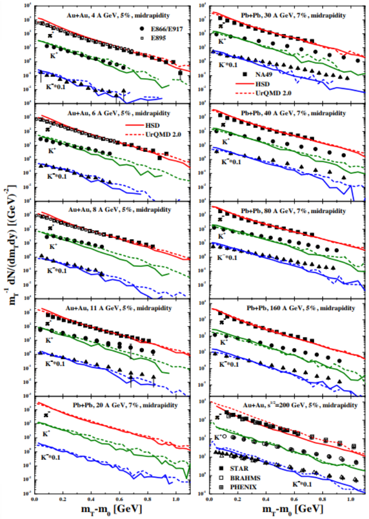
- The Ultra-relativistic Quantum Molecular Dynamics model starts with such an effective-solution of the relativistic BTE.
- The presence of an external potential results in an additional term on the LHS of eqn. (21).
- Hadrons and strings, excited in high-energy binary collisions, form the underlying degrees-of-freedom.
- Over 50 different baryon and 40 different meson species are included. It also has full particle-antiparticle symmetry, isospin-symmetry and only flavour-SU(3) states.
- During the stage I $\rightarrow$ II, transport $\rightarrow$ hydro. coupling; each particle is described by a Gaussian distribution of its total E, p and ρ_B.
- During the stage II $\rightarrow$ III, hydro. $\rightarrow$ transport coupling (or, freeze-out); all the fluid elements are transformed back to hadrons, which are then propagated via hadronic cascade.

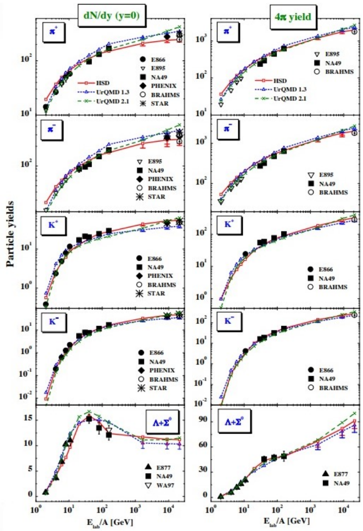
The UrQMD model: Simulation



Source: https://videos.cern.ch/record/1304862

The UrQMD model: Results





The UrQMD model: Afterword

- The results are obtained with a variant of the UrQMD called the hybrid-UrQMD—'hybrid' meaning that UrQMD has within it a hydro. simulation added to (or, coupled to) the transport simulation.
- Why do we need the hybrid?
  Because, pure-transport UrQMD is INCAPABLE of producing experimentally verifiable data. We have to use hydro. to match the UrQMD results with experiment.
- This proves that hydro. is actually one of the stages in the evolution of a HIC system.
- The schematic shown in slide 3 IS A CONJECTURE. We assume that the evolution has—at least—3 stages. The fact that the evolution actually, in reality, has a hydro. stage is PROVEN by the fact that pure-transport UrQMD is incapable of reproducing experimental data.
- So, hydro. HAS to exist $\implies$ particles at high temperatures & densities DO become fields $\implies$ these fields DO contribute to the QGP $\implies$ the QGP HAS to exist!

HIC evolution: Open questions

- We know, for a fact, that we are sending in particles. We also know, for a fact, that we get particles at the end. But, do we know for sure, whether the hydro. (field-dominated) stage is the ONLY stage of evolution in between the 2 transport (particle-dominated) stages?
  We don't! There might be more than one stage of evolution in between the first and last transport stages.
- Do we know EXACTLY WHEN the hydro. stage sets in? I.e., do we know exactly when the system thermalises?
  We don't! We assume that it thermalises, if a few criteria are satisfied. More interestingly, and recently, people have suggested that we could use hydro. in the absence of thermalisation—if certain criteria are satisfied. These are called the 'hydrodynamisation criteria'. So, we can apply hydro., but the system can still be outside thermal equilibrium.

QCD: Basics I

- A field is a physical quantity, expressed as a continuous function of position, x: $\phi(x)$.
- The Lagrangian density of a system of fields is a **functional**: $\mathcal{L}(\phi(x))$.
- The second stage of the evolution of a heavy-ion collision is considered to be a system of fields in thermal and dynamic equilibrium $\rightarrow$ it can be studied using hydrodynamics.
- Even in the non-dissipative (zeroth-order) case, the equations-of-motion of hydrodynamics form an incomplete set of equations, completed only by an equation-of-state (EOS) which encapsulates the relationship between the state variables of the system: $P \equiv P(n, \epsilon)$.
- In order to obtain the EOS, the thermodynamics of the system needs to be understood. For that, a knowledge of the partition function is required. For that, an understanding of the theory of strong interactions is needed.

  QCD is the preeminent theory of strong interactions.

QCD: Basics II:- The QCD Lagrangian

QCD Lagrangian:

$$\mathcal{L}_{QCD} = \overline{q}(i\gamma^{\mu} D_{\mu} - \mathcal{M})q - \frac{1}{4} F_{a}^{\mu\nu} F^{a}_{\mu\nu} \tag{24}$$

where

$$F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g f^{a}_{bc} A^{b}_{\mu} A^{c}_{\nu} \ , \tag{25}$$

with

$$\alpha_s(Q^2) = \frac{g^2}{4\pi} \approx \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda^2)} \tag{26}$$

- A_{μ} are gluons, q are quarks, $a = 1...8$, $\mathcal{M}$ is a phenomenological, colour-independent mass-matrix.
- For $Q^2 \gg \Lambda^2$, perturbative approximations are possible.
- For $Q^2 \sim \Lambda^2$, perturbative calculations diverge with $\alpha_s \to \infty$.

Therefore, effective Lagrangians; with some properties of QCD – which can be applicable in small-Q^2 scenarios – are needed.

QCD: Basics III:- Chiral symmetry & symmetry breaking I

$\mathcal{L}_{QCD}$ possesses the symmetries of the strong interaction:

- The $U(1)$ transf., $q(x) \to e^{-i\theta} q(x)$, results in the conservation of the baryon-no. current (Nöther's theorem) and of the baryon no., B.
- The chirality of an object, or system, prevents it from being congruent to its mirror-image after a rotation around the mirror axis.
- A vectorial, $q \to \exp(-i\Theta_V^a G_a)q \approx (1 - i\Theta_V^a G_a)q$ & an axial-vectorial transf., $q \to \exp(-i\gamma_5 \Theta_A^a G_a)q \approx (1 - i\gamma_5 \Theta_A^a G_a)q$ can be defined.
- $\mathcal{L}_{QCD}$, for $m_q = 0$, is symmetric under these transformations.
- Diagonalising $\mathcal{M}$ to obtain: $\mathcal{L}_{m_q} = -\bar{q} m_q q$, breaks chiral symmetry explicitly, but small quark masses $\to$ an approximate chiral symmetry.
- **Explicit SB:** caused by a term in $\mathcal{L}$ that renders $\mathcal{L} \neq \mathcal{L}'$ under a symmetry transf. $q \to q'$.

QCD: Basics IV:- Chiral symmetry & symmetry breaking II

- If chiral symmetry were an exact (or approx.) symmetry of QCD, it would cause a degeneracy between states of different parity, in vacuum. But, the vacuum has a wide range of hadronic states; with a mass hierarchy $\implies$ **the chiral symmetry of QCD is, most likely, spontaneously broken in the vacuum**.
- **Spontaneous SB:** Not manifest in $\mathcal{L}$ (*i.e.*, $\mathcal{L} = \mathcal{L}'$ under $q \to q'$). The, energetically, most favourable vacuum state is not symmetric under $q \to q'$.
- Goldstone's theorem: The spontaneous breaking of a continuous, global symmetry implies the existence of massless & spinless bosons.
- One can, therefore, have an effective theory that has:
  - a simplified Lagrangian; obtained by integrating out some d.o.f.'s of QCD; with spontaneous chiral-symmetry breaking (SCSB),
  - pions (low mass & 0 spin) being realised as Goldstone bosons,
  - a mass term for the pions, breaking the lagrangian's chiral-symmetry explicitly – called explicit CSB (ECSB) &
  - all three phase-transitions: nuclear-LG, chiral & deconfinement.

The LSM: Basics

The linear sigma model (LSM) Lagrangian for 2 quark flavours:

$$\mathcal{L}_{LSM} = \overline{q}\left[i\partial_\mu\gamma^\mu - g(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})\right]q + \frac{1}{2}\left[(\partial_\mu\vec{\pi})^2 + (\partial_\mu\sigma)^2\right] - U(\sigma,\vec{\pi}) \tag{27}$$

- $U(\sigma,\vec{\pi}) = U_{SCSB} + U_{ECSB}$.
- $U_{SCSB} = \frac{\lambda^2}{4}\left(\sigma^2 + \vec{\pi}^2 + v^2\right)^2$.
- $U_{ECSB} = -f_\pi m_\pi^2 \sigma$.
- f_π is the pion decay constant and λ is the coupling constant.
- $\mathcal{L}_{LSM}$ is invariant under $SU(2)_L \times SU(2)_R$ transf., if $U_{ECSB} = 0$.
- v is the vacuum expectation value of the chiral field $\Phi(\sigma,\vec{\pi}) = \mathrm{Tr}[\Sigma]$, $\Sigma = \sigma + i\vec{\tau}\cdot\vec{\pi}$; hence the name **linear** sigma model.

The LSM: Thermodynamic potential

Using eqn. (27), one can get:

$$\Omega(T,\mu_q) = -\frac{T\ln Z}{V} = U(\sigma,\vec{\pi}) + \Omega_{q\bar{q}}(T,\mu_q) \ ,$$

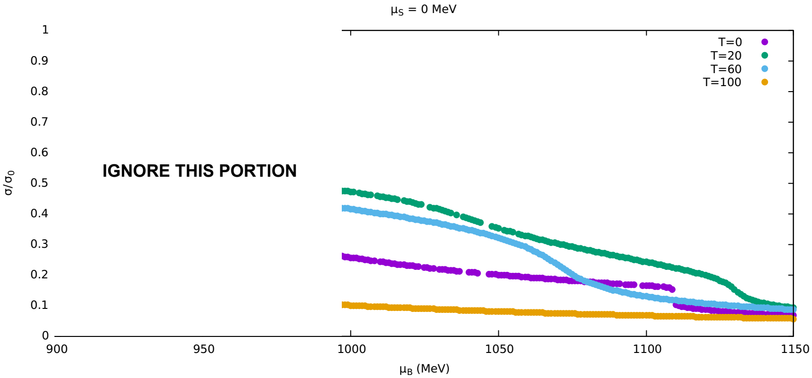
$$Z = \exp\left(-\frac{VU}{T}\right)\det_p\left\{[p_\mu\gamma^\mu + \mu_q\gamma^0 - g(\sigma + i\gamma_5\vec{\tau}\cdot\vec{\pi})]/T\right\} \ \&$$

$$\boxed{\Omega_{q\bar{q}} = -12\int\frac{d^3p}{(2\pi)^3}\left\{E + T\ln\left[1+e^{\left(\frac{\mu_q-E}{T}\right)}\right] + T\ln\left[1+e^{\left(\frac{-\mu_q-E}{T}\right)}\right]\right\}}$$

Here $Z \equiv$ the partition function, $V \equiv$ the volume of the system, Ω & $\Omega_{q\bar{q}} \equiv$ the thermodynamic potential and the quark-antiquark contribution, $T \equiv$ temperature and $\mu_q = \mu_B/3 \equiv$ the chemical potential.

All thermodynamic quantities, like ϵ, P & n can be obtained from Ω $\implies$ a parameter-dependent EOS: $P \equiv P(n,\epsilon)$ can be obtained.

QCD: Basics VI:- Phase-transitions

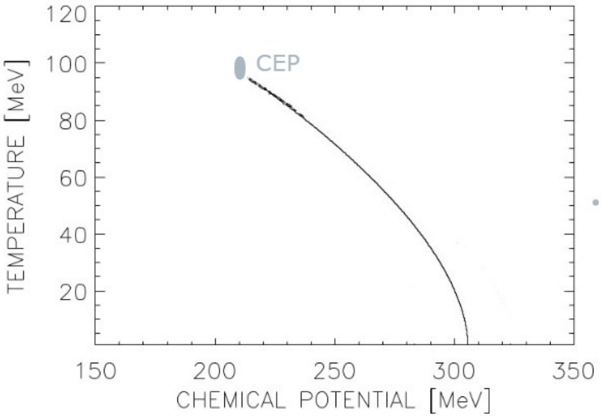


- $T = 0$ MeV $\rightarrow$ discontinuous jump $\implies$ first-order
- $T = 20$ & 60 MeV $\rightarrow$ continuous jumps $\implies$ crossover
- $T = 100$ MeV $\rightarrow$ no jump $\implies$ pure QGP, post transition
- Resulting ordered pairs (T, μ_B) are plotted as phase-boundaries

QCD: Basics V:- Order-parameters

- An order-parameter is a measure of the degree of order across the boundaries in a phase-transition system. It normally ranges between 0 in one phase and non-zero in the other. At the critical point, the order-parameter susceptibility will usually diverge $\rightarrow$ it goes through a continuous (for crossover) or a discontinuous (for first-order) jump in the presence of a phase-transition.
- From a theoretical perspective, order-parameters arise from symmetry breaking. When this happens, one needs to introduce one, or more, extra variables to describe the state of the system.
- Examples:
  - σ (the chiral-condensate) is the order-parameter for the chiral phase-transition. It goes from a finite value to 0 during a chiral phase-transition. In the chirally-asymmetric phase, it is non-zero. But, in the chirally-symmetric phase, it is 0.
  - Φ (the Polyakov loop) goes from 0 to 1 in the presence of a deconfinement transition. In the confined phase, the Polyakov loop is 0 and in the deconfined (quark) phase, it is 1.

The LSM: Phase diagram



Extending LSM to flavour-$SU(3)$ only changes the position of the critical end-point, not the nature of the first-order transition.

The LSM: Drawbacks

- There is no confinement-deconfinement mechanism. Therefore, the nucleons appear as bound states of three quarks.
- These bound-states need solving in a mean-field approximation, which breaks both the rotational- and the isospin-invariance of the theory and requires some projections onto physical states at the end.
- Apart from chiral symmetry, the model has not been connected directly to QCD.
- The model does not always give the correct phenomenology, e.g., the value of the isoscalar pion-nucleon scattering-length is too large.

The NJL model: Basics

In the simplest version of the Nambu–Jona-Lasinio (NJL) model, with only scalar and pseudo-scalar 4-fermion interaction terms:

$$\boxed{\mathcal{L}_{NJL} = \overline{q}\left(i\partial_\mu\gamma_\mu - m_0\right)q + \frac{G}{2}\left[(\overline{q}q)^2 + (\overline{q}i\gamma_5\vec{\tau}q)^2\right]} \qquad (28)$$

- $m_q = m_0 - G\langle\overline{q}q\rangle$.
- $\mathcal{L}_{NJL}$ is invariant under $SU(2)_L \times SU(2)_R$ transf., if $m_0 = 0$.
- The coupling constant G has the dimension $(\text{energy})^{-2}$, which makes the theory non-renormalisable and necessitates a 3-momentum cut-off to regularise divergent integrals.

The NJL model: Thermodynamic potential

Using eqn. (28), one can get:

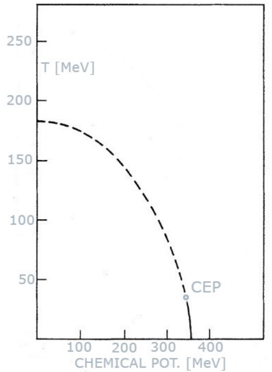
$$Z = \exp\left(-\frac{VG\langle\overline{q}q\rangle^2}{2T}\right)\det{}_p\left\{[p_\mu\gamma^\mu + \mu_q\gamma^0 - m_q]/T\right\}$$

and

$$\Omega = \frac{m_q - m_0}{2G} - 12\int\frac{d^3p}{(2\pi)^3}\left\{E + T\ln\left[1 + e^{\left(\frac{\mu_q - E}{T}\right)}\right] + T\ln\left[1 + e^{\left(\frac{-\mu_q - E}{T}\right)}\right]\right\}$$

All thermodynamic quantities, like ϵ, P & n can be obtained from Ω
$\Longrightarrow$ a parameter-dependent EOS: $P \equiv P(n, \epsilon)$ can be obtained.

The NJL model: Phase diagram



T_{CEP} ($= 50$ MeV) is lower than that obtained from LSM. The dashed line denotes a crossover transition.

The NJL model: Afterword

- Advantage:
  - NJL is a pre-QCD model for nucleons $\implies \mathcal{L}_{NJL}$ is constructed in a way that the symmetries of QCD are part and parcel of it.
- Disadvantages:
  - There is no confinement-deconfinement mechanism.
  - Quark interactions are point-like in character $\implies$ the theory is not renormalisable and requires a regularisation scheme.

The PNJL model: Basics

The LS and NJL models have quark d.o.f., but no deconfinement technique. Introducing the Polyakov loop as an approximate order-parameter for deconfinement, the NJL Lagrangian with a Polyakov-loop potential can be written as:

$$\boxed{\mathcal{L}_{PNJL} = \overline{q}\left(i\partial_\mu\gamma_\mu - m_0\right)q + \frac{G}{2}\left[(\overline{q}q)^2 + (\overline{q}i\gamma_5\vec{\tau}q)^2\right] - \mathcal{U}(\Phi, \Phi^*, T)} \tag{29}$$

- $\Phi = \frac{1}{3}\,\mathrm{Tr}[\exp\left(i\phi/T\right)]$ is the 3-colour-averaged, traced Polyakov-loop.
- A local, chirally-symmetric, scalar–pseudo-scalar, four-point interaction of the quark fields is introduced with an effective coupling strength G.

The PNJL model: Thermodynamic potential

The thermodynamic potential for the model reads:

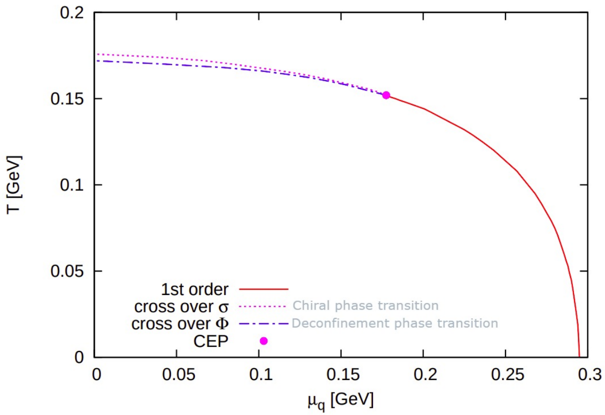
$$\Omega = \mathcal{U}(\Phi, \Phi^*, T) + g(\sigma, G) - \Omega_q \ ,$$

with

$$\begin{aligned}\Omega_q = 4 \int \frac{d^3p}{(2\pi)^3} \times \\ \Big\{ T \ln \Big[1 + 3\Phi e^{-(E_p - \mu_q^*)/T} + 3\Phi^* e^{-2(E_p - \mu_q^*)/T} + e^{-3(E_p - \mu_q^*)/T} \Big] \\ + T \ln \Big[1 + 3\Phi^* e^{-(E_p + \mu_q^*)/T} + 3\Phi e^{-2(E_p + \mu_q^*)/T} + e^{-3(E_p + \mu_q^*)/T} \Big] \\ + 3\Delta E_p \Theta(\Lambda^2 - \vec{p}^2) \Big\}\end{aligned}$$

where Φ goes from 0 to 1, signalling the deconfinement of quarks.

The PNJL model: Phase diagram



T_{CEP} ($= 150$ MeV) is higher than that obtained from NJL. The dashed lines denote the chiral and deconfinement crossover transitions.

Summary

- HIC's have both equilibrium and non-equilibrium stages of evolution.
- For the particle-dominated, non-equilibrium evolution, a microscopic transport approach is used to track the dynamics of the system.
- The transport approach requires a full, or effective, solution to the Boltzmann transport eqn., *e.g.*, UrQMD.
- UrQMD requires 'hybridisation' with hydro. to reproduce exp. results.
- The dynamics of the equilibrium-evolution is governed by hadronic and partonic field interactions, making hydrodynamics applicable.
- The EOS required to complete the set of hydro. equations-of-motion is provided by effective-field theories, which replicate the symmetries of QCD with a constructed, approximate, effective Lagrangian.
- Iterative changes to the effective models have provided insights into phenomena like chiral symmetry restoration and deconfinement.

SSB: Vacuum expectation value

Let us consider the Lagrangian:

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 \ . \qquad (30)$$

Remembering that:

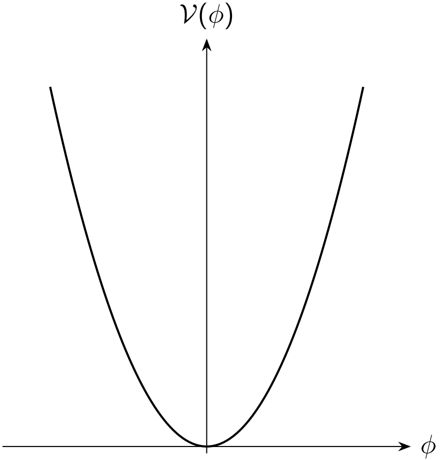
$$\mathcal{L}(\phi) = \mathcal{T}(\phi) - \mathcal{V}(\phi) \ , \qquad (31)$$

we have:

$$\mathcal{V}(\phi) = \frac{1}{2}m^2\phi^2 \ . \qquad (32)$$

Minimising $\mathcal{V}(\phi)$ w.r.t. ϕ, the minima, ϕ_0, gives the vacuum expectation value:

$$\phi_0 = v = 0 \ ; \qquad (33)$$
$$v \equiv \text{VEV of } \phi \ .$$



Thus, we get a(n) SH potential with a **single vacuum state**.

SSB: Degenerate vacuum states

Introducing the interaction-term $\frac{\lambda}{4}\phi^4$ & replacing m^2 with $-\mu^2$:

$$\mathcal{L}(\phi) = \frac{1}{2}\left(\partial_\mu \phi\right)^2 + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4}\phi^4 \ . \tag{34}$$

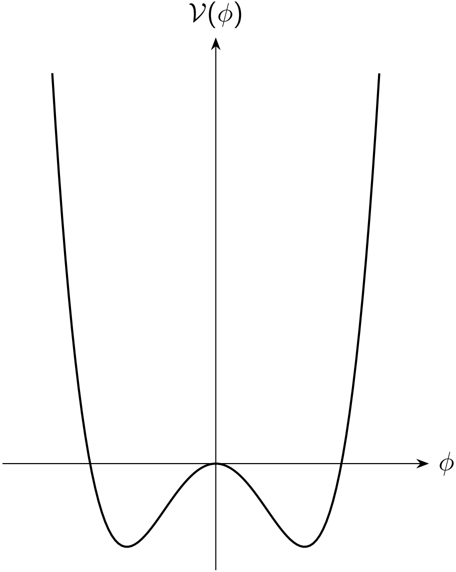
$\mathcal{L}$ is symmetric under the discrete transformation $\phi \to -\phi$ and:

$$\mathcal{V}(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}\phi^4 \tag{35}$$

has two minima:

$$\phi_0 = \pm v = \pm\frac{\mu}{\sqrt{\lambda}} \ . \tag{36}$$

Thus, the potential has **two, equally-likely, degenerate, symmetric vacuum states**.



SSB: Discrete symmetry breaking

Supposing the system is near (say) the +ve minima; we redefine ϕ as:

$$\phi(x) = v + \sigma(x) \; ; \qquad (37)$$

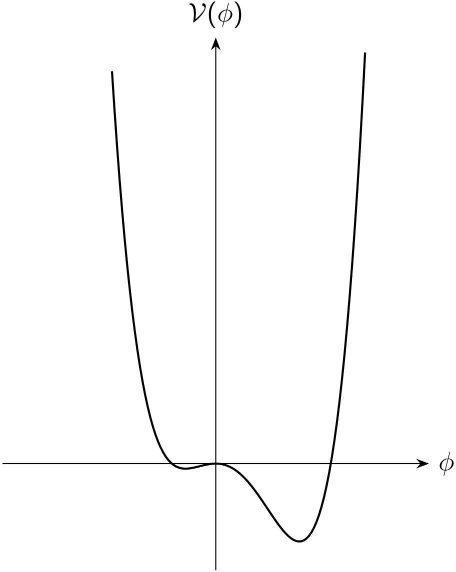
to find that:

$$\mathcal{L}(\sigma) = \frac{1}{2}\left(\partial_\mu \sigma\right)^2 + \frac{1}{2}\left(2m^2\right)\sigma^2 + \frac{m^2}{v}\sigma^3 - \frac{\lambda}{4}\sigma^4 \qquad (38)$$

is asymmetric under $\sigma \to -\sigma$ and:

$$\mathcal{V}(\sigma) = -\frac{1}{2}\left(2m^2\right)\sigma^2 - \frac{m^2}{v}\sigma^3 + \frac{\lambda}{4}\sigma^4 \qquad (39)$$

gives us **two, spontaneously symmetry-broken vacuum states**.



SSB: Continuous symmetry breaking

For a continuous transformation $\phi^i \to R^{ij}\phi^i$; with N real, scalar fields, $\phi^i(x)$; that leaves the classical, linear sigma model's Lagrangian:

$$\mathcal{L}(\phi^i) = \frac{1}{2}\left(\partial_\mu \phi^i\right)^2 + \frac{1}{2}\mu^2\left(\phi^i\right)^2 - \frac{\lambda}{4}\left(\phi^i\right)^4 \tag{40}$$

unchanged (*i.e.* $\mathcal{L}$ is symm. under an $O(N)$ rot.), the potential:

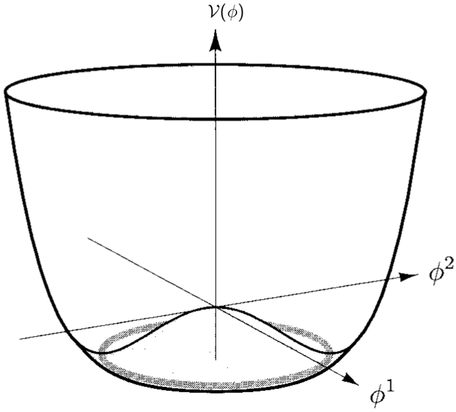
$$\mathcal{V}(\phi^i) = -\frac{1}{2}\mu^2\left(\phi^i\right)^2 + \frac{\lambda}{4}\left(\phi^i\right)^4 \tag{41}$$

is minimised for $\phi_0^i = v = \frac{\mu}{\sqrt{\lambda}}$; where v gives the length of the vector ϕ_0^i, while its direction is arbitrary. Choosing coordinates such that ϕ_0^i points in the N^{th} direction, $\phi_0^i = (0, 0, ..., 0, v)$, the shifted-fields, $\phi^i(x) = \left[\pi^k(x), v + \sigma(x)\right]$; with $k = 1, ..., N-1$; describe $\mathcal{L}$ as:

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\left(\partial_\mu \pi^k\right)^2 + \frac{1}{2}\left(\partial_\mu \sigma\right)^2 - \frac{1}{2}\left(2\mu^2\right)\sigma^2 - \sqrt{\lambda}\mu\sigma^3 \\ & - \sqrt{\lambda}\mu\left(\pi^k\right)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}\left(\pi^k\right)^2\sigma^2 - \frac{\lambda}{4}\left(\pi^k\right)^4 \quad .\end{aligned} \tag{42}$$

SSB: Goldstone's theorem

- The massive σ-field & the $N-1$, massless π-fields spontaneously break the original $O(N)$ symmetry.
- The remaining $O(N-1)$ sub-group rotates the π's amongst themselves.
- For $N=2$, $\mathcal{V}(\phi^i)$ becomes the sombrero-potential.
- Oscillations of ϕ^i along the potential's trough correspond to the π-fields.
- Oscillations in the radial direction correspond to the σ-field of mass $\sqrt{2}\mu$.



Goldstone's theorem: The spontaneous breaking of a continuous, global symmetry implies the existence of massless & spinless bosons.